

A note on the short-time quantum propagator

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Abstract

In the Feynman formalism of quantum mechanics one encounters a postulate, namely, that the propagator in an infinitesimal time-interval is the classical wave function. This postulate, which was later studied thoroughly by Holland, was recently highlighted by using the improved Makri-Miller propagator. The present note, whose conclusion is in agreement with that recent achievement, demonstrates that the Heisenberg picture of quantum mechanics invariably includes the Feynman postulate and is able to yield a proof for it. In other words, by starting out from the Heisenberg picture, it is proved that when terms of second-order in time can be neglected, the dynamics of a system is classical.

Keywords: Heisenberg picture; Short-time propagator; Quantum and classical dynamics.

1 Introduction

In the formulation of quantum mechanics from the Feynman point of view, there exists a postulate reading that, in an infinitesimal time-interval, the propagator is just the classical wave function [4,10]. It was then discussed by Holland [9] that along each infinitesimal interval, the motion of the particle is classical and that the full quantum mechanics at later times will be recovered by the superposition principle. Accordingly, it was recently shown by using an improved version of the Makri-Miller propagator that when terms of $\mathcal{O}(\Delta t^2)$ can be neglected, the classical trajectories arise from a short-time quantum propagator [12]. The same idea was also applied to generate the full quantum mechanics of the cold Bose atoms around a crossing of quantum waveguides with different geometries using the Chapman-Kolmogorov identity [13]. Both the proof and its applications seem to be crucial for the foundation of quantum mechanics because they shed light on one of the famous Oxford Questions [11] which reads: “Does the classical world emerge from the quantum, and if so which concepts are needed to describe this emergence?”

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It is commonly stated that the classical level arises when \hbar (Planck's constant) tends to zero. This viewpoint can at first sight be realized by a fundamental equation of quantum mechanics (as Dirac called it [1]) which represents a mapping from the commutation of two dynamical variables into their Poisson bracket; that is,

$$[u, v] = i\hbar \{u, v\}, \quad (1)$$

where u and v are arbitrary functions of a set of canonical coordinates and momenta [14]. This is because for a non-zero Poisson bracket when $\hbar \rightarrow 0$, u and v commute and therefore become classical dynamical variables. This approach, however, did not satisfy Bohm and hence he reformulated the problem by means of invoking different orders of time and also the concept of "quantum potential" [15].

The question to be addressed here is as follows: How does the Heisenberg picture of quantum mechanics handle the Feynman postulate? That is to say, can the Heisenberg picture present a proof for the Feynman postulate, or should the statement in this picture be in the form of an axiom? The purpose of the present note is to demonstrate that it is actually possible to start out from the Heisenberg picture of quantum mechanics and from there to prove the Feynman postulate. More precisely, the following statement will be offered: The Heisenberg picture inherently includes the notion that, when terms of $\mathcal{O}(\Delta t^2)$ can be neglected, the dynamics of a system is classical. The next Section is dedicated to the proof.

2 Heisenberg picture and Feynman postulate

The Heisenberg picture of quantum mechanics supposes that the dynamical variables are matrices and hence do not satisfy the commutative axiom of multiplication. Those dynamical variables vary with time according to Heisenberg's equation of motion

$$i\hbar \frac{d\hat{\xi}}{dt} = [\hat{\xi}, \hat{H}], \quad (2)$$

where $\hat{\xi}$ denotes a dynamical variable and \hat{H} , corresponding to the total energy in the Heisenberg picture, denotes a linear operator which is just the transform of the Hamiltonian operator occurring in the Schrödinger picture. The relation of the given dynamical variable when written either in the Heisenberg or the Schrödinger picture is given by the following unitary transformation [3,6]:

$$\hat{\xi}_H(t) = e^{i\hat{H}\frac{(t-t_0)}{\hbar}} \hat{\xi} e^{-i\hat{H}\frac{(t-t_0)}{\hbar}} \quad (3)$$

Given the position \hat{q} and momentum \hat{p} as two dynamical variables, one can thus by (3) find the corresponding Heisenberg representations. Let $|\psi(t_0)\rangle$ denote the state of a dynamical system at time t_0 and define the following representations for position and momentum in the Heisenberg picture:

$$\langle \psi(t_0) | \hat{q}_H(t) | \psi(t_0) \rangle \equiv Q(t) \quad (4)$$

$$\langle \psi(t_0) | \hat{p}_H(t) | \psi(t_0) \rangle \equiv P(t) \quad (5)$$

Let $\Delta t = t - t_0$ denote a time interval. One can then with the help of (4) and (5) and also using Hadamard's lemma [5,7] write:

$$\begin{aligned} & Q(t_0 + \Delta t) \\ &= \left\langle \psi(t_0) \left| \hat{q}_H(t_0) + \frac{i}{\hbar} [\hat{H}, \hat{q}] \Delta t + \left(\frac{i}{\hbar} \right)^2 [\hat{H}, [\hat{H}, \hat{q}]] \frac{\Delta t^2}{2} + \mathcal{O}(\Delta t^3) \right| \psi(t_0) \right\rangle \\ &= Q(t_0) + \left\langle \psi(t_0) \left| \frac{i}{\hbar} [\hat{H}, \hat{q}] \right| \psi(t_0) \right\rangle \Delta t + \mathcal{O}(\Delta t^2); \end{aligned} \quad (6)$$

$$\begin{aligned} & P(t_0 + \Delta t) \\ &= \left\langle \psi(t_0) \left| \hat{p}_H(t_0) + \frac{i}{\hbar} [\hat{H}, \hat{p}] \Delta t + \left(\frac{i}{\hbar} \right)^2 [\hat{H}, [\hat{H}, \hat{p}]] \frac{\Delta t^2}{2} + \mathcal{O}(\Delta t^3) \right| \psi(t_0) \right\rangle \\ &= P(t_0) + \left\langle \psi(t_0) \left| \frac{i}{\hbar} [\hat{H}, \hat{p}] \right| \psi(t_0) \right\rangle \Delta t + \mathcal{O}(\Delta t^2). \end{aligned} \quad (7)$$

By assuming an infinitesimal time-interval, that is to say, $\Delta t \rightarrow 0$, and neglecting the terms of order $\mathcal{O}(\Delta t^2)$ in (6) and (7), these equations yield:

$$\lim_{\Delta t \rightarrow 0} \frac{Q(t_0 + \Delta t) - Q(t_0)}{\Delta t} = \dot{Q} = \left\langle \psi(t_0) \left| \frac{i}{\hbar} [\hat{H}, \hat{q}] \right| \psi(t_0) \right\rangle \quad (8)$$

$$\lim_{\Delta t \rightarrow 0} \frac{P(t_0 + \Delta t) - P(t_0)}{\Delta t} = \dot{P} = \left\langle \psi(t_0) \left| \frac{i}{\hbar} [\hat{H}, \hat{p}] \right| \psi(t_0) \right\rangle \quad (9)$$

One can with the help of Ehrenfest theorem [2] and the definitions (4) and (5) write:

$$\dot{Q} = \frac{d}{dt} \langle \psi(t_0) | \hat{q}_H(t) | \psi(t_0) \rangle = \frac{i}{\hbar} \left\langle \psi(t_0) \left| [\hat{H}, \hat{q}_H] \right| \psi(t_0) \right\rangle \quad (10)$$

$$\dot{P} = \frac{d}{dt} \langle \psi(t_0) | \hat{p}_H(t) | \psi(t_0) \rangle = \frac{i}{\hbar} \left\langle \psi(t_0) \left| [\hat{H}, \hat{p}_H] \right| \psi(t_0) \right\rangle \quad (11)$$

The right-hand sides of (10) and (11) should respectively be equal to the right-hand sides of (8) and (9); hence:

$$\frac{i}{\hbar} \left\langle \psi(t_0) \left| [\hat{H}, \hat{q}] \right| \psi(t_0) \right\rangle = \frac{i}{\hbar} \left\langle \psi(t_0) \left| [\hat{H}, \hat{q}_H] \right| \psi(t_0) \right\rangle \quad (12)$$

$$\frac{i}{\hbar} \left\langle \psi(t_0) \left| \left[\hat{H}, \hat{p} \right] \right| \psi(t_0) \right\rangle = \frac{i}{\hbar} \left\langle \psi(t_0) \left| \left[\hat{H}, \hat{p}_H \right] \right| \psi(t_0) \right\rangle \quad (13)$$

Now with the help of (12) and (13) and using (1), (8) and (9) can respectively be rewritten as

$$\dot{Q} = \{Q, \mathcal{H}\}, \quad (14)$$

and

$$\dot{P} = \{P, \mathcal{H}\}, \quad (15)$$

where \mathcal{H} denotes the corresponding classical Hamiltonian due to the Poisson bracket formalism. Equations (14) and (15) are the classical equations of motion for two variables P and Q , defined by the representations (4) and (5), in an infinitesimal time-interval when one has neglected terms of $\mathcal{O}(\Delta t^2)$.

Consequently, when starting out from the Heisenberg picture, terms of the second-order in time can be neglected, the dynamics of a system becomes classical. This proves the Feynman postulate and thereby fulfills the main aim of this Section.

3 Concluding remark

The Feynman formalism of quantum mechanics contains a famous postulate stating that during an infinitesimal time-interval, the propagator is the classical wave function. This postulate, which was later analyzed by Holland, was recently highlighted by de Gosson and Hiley by using an improved version of the Makri-Miller propagator. The question addressed in the present note was whether the Heisenberg picture of quantum mechanics can directly handle the Feynman postulate and allow a proof for that. It was indeed demonstrated that the Heisenberg picture inherently includes the Feynman postulate and supports a proof for the following general statement: “When terms of $\mathcal{O}(\Delta t^2)$ can be neglected, the dynamics of a system is classical.” This conclusion is in agreement with the recent de Gosson-Hiley achievement [12]—a result derived from the Bohmian trajectories point of view.

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- [14] The coefficient \hbar in (1) was first considered as an assumption in [1], but later, Dirac presented a more rigorous justification in [6].
- [15] He argued convincingly that h is considered as a fixed number and therefore it is not mathematically sensible to talk about a limit value for it (e.g., see [8]).